One more limit of Lalescu kind sequence.

https://www.linkedin.com/feed/update/urn:li:activity:6831821081725194240

Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim\limits_{n\to\infty} \dfrac{a_n}{n!} = a > 0$.

Find $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$.

Solution by Arkady Alt, **San Jose**,**California**, **USA**.

Let $b_n := \dfrac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}$. Since $\lim\limits_{n\to\infty}\sqrt[n]{\dfrac{a_n}{n!}} = 1$ (because $\dfrac{a_n}{n!}$ is bounded) and

$\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$ then $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{a_n}}{n} = \lim\limits_{n\to\infty}\left(\sqrt[n]{\dfrac{a_n}{n!}}\cdot\dfrac{\sqrt[n]{n!}}{n}\right) = \lim\limits_{n\to\infty}\sqrt[n]{\dfrac{a_n}{n!}}\cdot\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$.

and, therefore, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\left(\dfrac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\dfrac{n}{\sqrt[n]{a_n}}\cdot\dfrac{n+1}{n}\right) = 1$.

Hence, $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = \lim\limits_{n\to\infty}\left(\dfrac{\sqrt[n]{a_n}}{n}\cdot\dfrac{b_n-1}{\ln b_n}\cdot n\ln b_n\right) = \dfrac{1}{e}\lim\limits_{n\to\infty} n\ln b_n =$

$\dfrac{1}{e}\ln\left(\lim\limits_{n\to\infty} b_n^n\right) = \dfrac{1}{e}\ln\left(\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}\cdot\dfrac{1}{\sqrt[n+1]{a_{n+1}}}\right) = \dfrac{1}{e}\ln\left(\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{(n+1)!}\cdot\dfrac{n!}{a_n}\cdot\dfrac{n+1}{\sqrt[n+1]{a_{n+1}}}\right)\right) =$

$\dfrac{1}{e}\ln\left(\lim\limits_{n\to\infty}\dfrac{n+1}{\sqrt[n+1]{a_{n+1}}}\right) = \dfrac{1}{e}\ln e = \dfrac{1}{e}$.